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Solutions of the f–g field equations

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Abstract. The f–g field equations are solved for an arbitrary mixing term. Several techniques are given to generate solutions of f–g field equations with and without sources for a special mixing term. Some new solutions are also presented.

1. Introduction

Analogous to the ρ–γ model of hadron electrodynamics (Lee *et al* 1967), Isham *et al* (1971) have developed a theory of strong and weak gravitations, the f–g field theory, to describe the mixing of the gravitation with a massive spin-two f-meson which interacts universally with hadrons through the energy momentum tensor. In the f–g field theory the gravitation interacts directly with leptons, but only indirectly with hadronic matter. In this unification scheme of gravitational and strong nuclear forces it is assumed that the interaction term in the total Lagrangian density is free of derivatives and reduces to the form of a Pauli–Fierz mass term in the flat space approximation.

These assumptions do not determine the mixing term uniquely. There are infinitely many ways to satisfy these requirements. Two different examples have been used so far (Isham *et al* 1971, Isham and Storey 1978). In spite of this defect, it has been observed that the classical source-free field equations corresponding to different mixing terms have common solutions (Salam and Strathdee 1978). One of the purposes of this paper is to clarify this point. Irrespective of the form of the mixing term, we show that there exists a simple relation between the Einstein tensors of the f and g fields. By using this remarkable property, it is possible to state that if one of the fields is a metric of an Einstein space then the other field is also a metric of a different Einstein space (Gürses 1980). Since this statement is independent of the mixing term, Salam and Strathdee (1977, 1978) and Isham and Storey (1978) solutions are also solutions of the f–g field equations with an unspecified interaction Lagrangian density.

The free Lagrangian density contains two Einstein parts corresponding to the fields f and g, respectively, in addition to the mixing term. Hence the field equations of the f–g field theory are more complicated than the Einstein field equations. In order to simplify the field equations the symmetries of the space–times have been increased and hence some few solutions have been found. The second purpose of this work is to propose a method of generating solutions of the source-free f–g field equations from Einstein

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spaces. We also divide the solutions into two classes according to whether the f and g fields are isometric or not.

It has been suggested that the classical f–g solutions may provide a mechanism for quark confinement (Salam and Strathdee 1977a, 1978). In this connection some spherically symmetric solutions have been found and interpreted as the effective tensorial potential for confinement. To support this, the massive Klein–Gordon equation (which is assumed to be the equation of motion of the quark field) has been solved and it was observed that the energy spectrum is the same as that for a harmonic oscillator potential. Another purpose of this paper is to improve this program a little bit further. In their model, Salam and Strathdee neglected both the leptonic and hadronic matter. In this work, we shall assume the existence of hadronic matter but neglect leptonic matter. We shall find some solutions when the hadronic energy momentum tensor is of the form of electromagnetic and Yang–Mills energy momentum tensors.

As a summary, this work contains: (i) a relation between the Einstein tensors of the f and g fields (f–g identities) in § 2, (ii) solution generation techniques and source-free solutions in § 3, (iii) f–g field equations with hadronic matter and solutions in § 4 and (iv) some matrix identities and applications of the solution generation techniques presented in § 3 are given in the appendices 1 and 2, respectively.

Our notation and conventions are as follows: $f_{\mu\nu}$ and $g_{\mu\nu}$ are the covariant components of the tensor fields f and g, and their inverses are $f^{\mu\nu}$ and $g^{\mu\nu}$ respectively. Also

$$f = \det f_{\mu\nu} \qquad g = \det g_{\mu\nu}.$$

Geometrical quantities, such as the Ricci tensors $R_{\mu\nu}(f)$ and $R_{\mu\nu}(g)$ corresponding to the fields f and g respectively, are constructed by defining two symmetric connections $\Gamma^{\alpha}_{\mu\nu}(f)$ and $\Gamma^{\alpha}_{\mu\nu}(g)$ from f and g respectively, as in Riemannian geometry. We shall fix only the signature of the g field as -2. Later we shall see that the signature of the f field can be different from that of the g field. The components of the curvature tensor and its contractions in terms of the affine connection are given by

$$R^{\alpha}_{\ \mu\beta\nu} = \Gamma^{\alpha}_{\mu\nu,\beta} - \Gamma^{\alpha}_{\mu\beta,\nu} + \Gamma^{\alpha}_{\sigma\beta}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\mu\beta}$$

$$R_{\mu\nu} = R^{\alpha}_{\ \mu\alpha\nu}.$$

The Ricci scalars are

$$R(f) = f^{\mu\nu}R_{\mu\nu}(f) \qquad R(g) = g^{\mu\nu}R_{\mu\nu}(g).$$

2. The f–g identities

The essential prescription in the f–g theory is that the hadronic matter Lagrangian is to be formed using $f^{\mu\nu}$ as a metric tensor while for the leptonic one must use $g^{\mu\nu}$. Thus the combined Lagrangian is

$$\mathscr{L} = \mathscr{L}_f + \mathscr{L}_g + \mathscr{L}_{f-g} \tag{2.1}$$

where

$$\mathscr{L}_f = -K_f^{-2}(-f)^{1/2}R(f) + \mathscr{L}(\text{hadrons}, f) \tag{2.2}$$

$$\mathscr{L}_g = -K_g^{-2}(-g)^{1/2}R(g) + \mathscr{L}(\text{leptons}, g). \tag{2.3}$$

The hadronic and the leptonic parts of the universe interact via an f–g mixing $\mathcal{L}_{f-g}$. This term is required to have the following properties (Isham *et al* 1971, Salam and Strathdee 1978).

(i) In addition to being a scalar density it contains no derivative terms.

(ii) It leads to the emergence of mass of the f field, corresponding to the finite-range nature of strong interactions, whereas the g field remains massless, corresponding to the infinite range of gravitational interactions.

With the first assumption the general form of the mixing term may be written as

$$\mathcal{L}_{f\cdot g} = \sqrt{-g}\tau \tag{2.4}$$

where τ is a scalar with the functional form

$$\tau = \tau(f_{\mu\nu}, g_{\mu\nu}, \phi^{\mu}{}_{\nu}, \rho^{\mu}{}_{\nu}, \Phi_A) \tag{2.5}$$

where

$$\phi^{\mu}{}_{\nu} = f^{\mu\alpha}g_{\alpha\nu} \qquad \rho^{\mu}{}_{\nu} = g^{\mu\alpha}f_{\alpha\nu} \tag{2.6}$$

and Φ_A, $A = 1, \ldots, N$, are the fields other than the f and g fields, N being the number of extra fields. The presence of these other fields in the interaction Lagrangian density makes the field equations much more complicated. There are several cases where these terms cannot be separated from $\mathcal{L}_{f-g}$ and absorbed in one of the free parts $\mathcal{L}_f$ or $\mathcal{L}_g$. An example is

$$\tau = f^{\mu\nu}g^{\alpha\beta}E_{\mu\alpha}E_{\nu\beta} \tag{2.7}$$

where $E_{\mu\nu}$ are the components of the electromagnetic field tensor. In this work we shall be interested only in those mixing terms that contain only the f and g fields. With this assumption (2.5) takes a very simple form

$$\tau = \tau(\phi^{\mu}{}_{\nu}) \tag{2.8}$$

where the functional dependence on $\rho^{\mu}{}_{\nu}$ is omitted because this mixed tensor is the inverse of $\phi^{\mu}{}_{\nu}$, i.e.

$$\phi^{\mu}{}_{\alpha}\rho^{\alpha}{}_{\nu} = \rho^{\mu}{}_{\alpha}\phi^{\alpha}{}_{\nu} = \delta^{\mu}{}_{\nu}. \tag{2.9}$$

In (2.4) we could use $\sqrt{-f}$ instead of $\sqrt{-g}$ to make $\mathcal{L}_{f-g}$ a scalar density of the correct weight. This would not make any difference, because

$$\begin{aligned}\mathcal{L}_{f-g} &= \sqrt{-f}\tau = \sqrt{-g}\sqrt{f/g}\tau \\ &= \sqrt{-g}\,(\det \phi^{\alpha}{}_{\beta})^{-1/2}\tau\end{aligned}$$

but $\det(\phi^{\alpha}{}_{\beta})$ is also a function of $\phi^{\mu}{}_{\nu}$, hence we can always write that

$$\mathcal{L}_{f-g} = \sqrt{-g}\tau(\phi^{\mu}{}_{\nu}). \tag{2.10}$$

The mass term for the f field is produced by letting (Isham *et al* 1971)

$$\begin{aligned}f^{\mu\nu} &= \eta^{\mu\nu} + K_f F^{\mu\nu} \\ g^{\mu\nu} &= \eta^{\mu\nu} + K_g H^{\mu\nu}\end{aligned}$$

and expanding $\mathcal{L}_{f-g}$ up to quadratic terms and equating it to the Pauli–Fierz Lagrangian density

$$-\tfrac{1}{4}M^2[F_{\mu\nu}F_{\alpha\beta}\eta^{\mu\alpha}\eta^{\nu\beta} - (F_{\alpha\beta}\eta^{\alpha\beta})^2] \tag{2.11}$$

where $\eta^{\mu\nu}$ is the Minkowski metric. We may write

$$\mathscr{L}_{f-g} \simeq \mathscr{L}^0 + \mathscr{L}^{\mathrm{I}} + \mathscr{L}^{\mathrm{II}}$$

where $\mathscr{L}^0$, $\mathscr{L}^{\mathrm{I}}$ and $\mathscr{L}^{\mathrm{II}}$ contain no terms, linear terms, and quadratic terms of the F and H fields, respectively. In order to have a mass term for the F field we must let $\mathscr{L}^0 = \mathscr{L}^{\mathrm{I}} = 0$ and equate $\mathscr{L}^{\mathrm{II}}$ to (2.11). Since they are neglected, the higher-order terms are left free. Hence, the two properties given at the beginning of this section are not enough to fix the mixing term $\mathscr{L}_{f-g}$.

The most commonly used mixing terms are

$$\mathscr{L}^{\mathrm{I}}_{f-g} = -\frac{M^2}{4K_f^2}\sqrt{-g}(f/g)^{1/2-u}(f^{\mu\nu}-g^{\mu\nu})(f^{\alpha\beta}-g^{\alpha\beta})(g_{\mu\alpha}g_{\nu\beta}-g_{\mu\nu}g_{\alpha\beta}) \tag{2.12}$$

where in addition to the mass M an arbitrary parameter u has been introduced, and K_f is the strong analogue of the gravitational constant K_g. $\mathscr{L}^{\mathrm{I}}{}_{f-g}$ may be given in terms of $\phi^{\mu}{}_{\nu}$ so that τ takes the form

$$\tau^{\mathrm{I}} = -\frac{M^2}{4K_f^2}(\det\phi^{\alpha}{}_{\beta})^{u-1/2}[\phi^{\sigma}{}_{\mu}\phi^{\mu}{}_{\sigma} - (\phi^{\mu}{}_{\mu})^2 + 6\phi^{\mu}{}_{\mu} - 12]. \tag{2.13}$$

Alternatively,

$$\mathscr{L}^{\mathrm{II}}_{f-g} = \lambda\sqrt{-g} + \lambda'\sqrt{-f} - (\lambda+\lambda')(-f)^{\alpha}(-g)^{\beta}\{-\det[xg^{\mu\nu} + (1-x)f^{\mu\nu}]\}^{\alpha+\beta-1/2} \tag{2.14}$$

where the parameters are subject to two constraints

$$2[-\alpha x + \beta(1-x)](\lambda+\lambda') = -x\lambda' + (1-x)\lambda$$
$$(\alpha+\beta-\tfrac{1}{2})x(x-1)(\lambda+\lambda')^2 = \tfrac{1}{4}\lambda\lambda'.$$

The scalar function τ for this case is given by

$$\tau^{\mathrm{II}} = \lambda + \lambda'(\det\phi^{\mu}{}_{\nu})^{-1/2} - (\lambda+\lambda')(\det\phi^{\mu}{}_{\nu})^{\alpha}\{-\det[x\delta^{\mu}{}_{\nu} + (1-x)\phi^{\mu}{}_{\nu}]\}^{\alpha+\beta-1/2}. \tag{2.15}$$

The field equations corresponding to a general coupling term with $\tau = \tau(\phi^{\mu}{}_{\nu})$ are found as follows. Upon varying $f^{\mu\nu}$, the action principle $\delta\int \mathscr{L}\,\mathrm{d}^4x = 0$ gives the field equations

$$G_{\mu\nu}(f) = R_{\mu\nu}(f) - \tfrac{1}{2}f_{\mu\nu}R(f) = K_f^2 T_{\mu\nu}{}^{f} \tag{2.16}$$

where

$$T_{\mu\nu}{}^{f} = \frac{1}{\sqrt{-f}}\frac{\partial\mathscr{L}_{fg}}{\partial f^{\mu\nu}} + T_{\mu\nu}(f, \text{hadrons}). \tag{2.17}$$

Variation of $g^{\mu\nu}$ yields the Einstein equations

$$G_{\mu\nu}(g) = R_{\mu\nu}(g) - \tfrac{1}{2}g_{\mu\nu}R(g) = K_g{}^2 T_{\mu\nu}{}^{g} \tag{2.18}$$

where

$$T_{\mu\nu}{}^{g} = \frac{1}{\sqrt{-g}}\frac{\partial\mathscr{L}_{f-g}}{\partial g^{\mu\nu}} + T_{\mu\nu}(g, \text{leptons}). \tag{2.19}$$

We define a mixed tensor

$$t^{\mu}{}_{\nu} = \frac{\partial\tau}{\partial\phi^{\nu}{}_{\mu}} \tag{2.20}$$

which has the following properties

$$t^{\beta}{}_{\mu}g_{\beta\nu}=t^{\beta}{}_{\nu}g_{\beta\mu} \tag{2.21}$$

$$f^{\beta\mu}t^{\nu}{}_{\beta}=f^{\beta\nu}t^{\mu}{}_{\beta}. \tag{2.22}$$

The proof of these identities is as follows. Since the scalar function τ is a polynomial of the invariants of the mixed tensor $\phi^{\mu}{}_{\nu}$, then $t^{\mu}{}_{\nu}$ is a tensorial function of $\phi^{\mu}{}_{\nu}$; i.e.

$$t^{\mu}{}_{\nu}=\sum_i t_i \underbrace{\phi^{\mu}{}_{\alpha}\phi^{\alpha}{}_{\rho}\ldots\phi^{\rho}{}_{\nu}}_{i\ \text{times}} \tag{2.23}$$

where t_i are scalars, then the proofs of (2.21) and (2.22) become obvious. Utilising these identities we find

$$g^{\mu\alpha}\frac{\partial\tau}{\partial g^{\alpha\nu}}+f^{\mu\alpha}\frac{\partial\tau}{\partial f^{\alpha\nu}}=0 \tag{2.24}$$

which leads to an important relation

$$g^{\mu\alpha}\frac{\partial\mathcal{L}_{fg}}{\partial g^{\alpha\nu}}+f^{\mu\alpha}\frac{\partial\mathcal{L}_{f\cdot g}}{\partial f^{\alpha\nu}}=-\tfrac{1}{2}\delta^{\mu}{}_{\nu}\mathcal{L}_{fg} \tag{2.25}$$

where

$$\frac{\partial\mathcal{L}_{fg}}{\partial g^{\mu\nu}}=-\tfrac{1}{2}g_{\mu\nu}\mathcal{L}_{fg}-\sqrt{-g}\,t^{\beta}{}_{\alpha}\phi^{\alpha}{}_{\mu}g_{\beta\nu} \tag{2.26}$$

$$\frac{\partial\mathcal{L}_{fg}}{\partial f^{\mu\nu}}=\sqrt{-g}\,t^{\beta}{}_{\mu}g_{\beta\nu}. \tag{2.27}$$

Equation (2.25) is also equivalent to

$$\sqrt{-g}\,T^{g\mu}{}_{\nu}+\sqrt{-f}\,T^{f\mu}{}_{\nu}=-\tfrac{1}{2}\mathcal{L}_{f\text{-}g}\delta^{\mu}{}_{\nu}+\sqrt{-g}\,T^{\mu}{}_{\nu}(g,\text{ leptons})+\sqrt{-f}\,T^{\mu}{}_{\nu}(f,\text{ hadrons}) \tag{2.28}$$

which means that the total energy momentum tensor density of the combined system has a pressure-like term in addition to the leptonic and hadronic energy momentum tensor densities. This so-called 'bag' term should play an important role in the confinement problem. In the case of no source (2.28) reduces to

$$K_g^{-2}\sqrt{-g}\,G^{\mu}{}_{\nu}(g)+K_f^{-2}\sqrt{-f}\,G^{\mu}{}_{\nu}(f)=-\tfrac{1}{2}\mathcal{L}_{f\text{-}g}\delta^{\mu}{}_{\nu}. \tag{2.29}$$

We refer to these relations as the *f–g* identities.

An immediate consequence of (2.29) is the generalisation of a result reported recently (Gürses 1980). If $g_{\mu\nu}$ is an Einstein space

$$G_{\mu\nu}(g)=\lambda_g g_{\mu\nu} \qquad \lambda_g=\text{constant} \tag{2.30}$$

then $f_{\mu\nu}$ is also an Einstein space

$$G_{\mu\nu}(f)=\lambda_f f_{\mu\nu} \qquad \lambda_f=\text{constant} \tag{2.31}$$

where

$$K_g^{-2}\lambda_g\sqrt{-g}+K_f^{-2}\lambda_f\sqrt{-f}=-\tfrac{1}{2}\mathcal{L}_{fg} \tag{2.32}$$

and vice versa. Salam and Strathdee (1978) have observed that the empty *f–g* field equations have sometimes common solutions, although their mixing terms are different. Our last result clarifies this point, since it shows that Einstein space solutions are independent of the mixing terms.

3. Solutions of source-free field equations

In this section we shall only consider the mixing term $\mathscr{L}^{\mathrm{I}}_{f\text{-}g}$, given in (2.12), and introduce some methods to solve the empty f–g field equations. Using (A5), (2.16) and (2.18) we obtain

$$G_{\mu\nu}(f)=\tfrac{1}{4}M^2(g/f)^u[-(u-\tfrac{1}{2})If_{\mu\nu}+2(3\phi-3+H)g_{\mu\nu}-2H_{\mu\nu}] \tag{3.1}$$

and

$$G_{\mu\nu}(g)=\tfrac{1}{4}\mu_0{}^2M^2(g/f)^{u-1/2}\{[-2\phi(3\phi-3+H)+uI]g_{\mu\nu} -2(2\phi-3+H)H_{\mu\nu}+2H^{\beta}{}_{\mu}H_{\beta\nu}\} \tag{3.2}$$

where

$$I=-4(\phi-1)(3\phi-3+H)-H(2\phi-2+H)+H^{\alpha\beta}H_{\alpha\beta} \tag{3.3}$$

$$\mu_0=K_g/K_f. \tag{3.4}$$

We shall now state a theorem concerning the above empty field equations (Gürses 1980).

Theorem. If $f^{\mu\nu}$ and $g^{\mu\nu}$ satisfy the empty f–g field equations (3.1) and (3.2) and if ϕ and $H^{\mu\nu}$ obey (A5) respectively, then the following three statements are equivalent

$$G_{\mu\nu}(f)=\lambda_f f_{\mu\nu} \tag{3.5a}$$

$$G_{\mu\nu}(g)=\lambda_g g_{\mu\nu} \tag{3.5b}$$

$$H^{\mu}{}_{\alpha}H^{\alpha}{}_{\nu}=aH^{\mu}{}_{\nu}+b\delta^{\mu}{}_{\nu} \tag{3.5c}$$

$$a=2\phi-3+H \tag{3.6}$$

where λ_f and λ_g are cosmological constants corresponding to the fields f and g respectively, and a and b are scalars. Without (3.5c) this theorem is a special case of the result given at the end of the previous section. For this special case, (3.5c) follows from (3.2) with

$$b=\mu_0^{-2}(4/M^2)(f/g)^{u-1/2}\lambda_g+2\phi(3\phi-3+H)-uI \tag{3.7}$$

and a is given in (3.6). Using (3.5)–(3.7) in (A7) we find

$$f_{\mu\nu}=(\phi^2+\phi a-b)^{-1}[(\phi+a)g_{\mu\nu}-H_{\mu\nu}] \tag{3.8}$$

$$g/f=(b-\phi^2-a\phi)(-\phi^2+\tfrac{3}{2}aH+b+a\phi-\tfrac{1}{2}H^2-a^2-\phi H) \tag{3.9}$$

and

$$I=-12(\phi-1)^2-b(\phi-1)H-H^2+aH+4b. \tag{3.10}$$

It follows by a straightforward calculation, from (3.6), (3.8) and (3.1), that

$$G^{\mu}{}_{\nu}(f)=\lambda_f f_{\mu\nu} \tag{3.11}$$

where

$$\lambda_f=\tfrac{1}{4}M^2(g/f)^u[(u-\tfrac{1}{2})I+2(\phi^2+a\phi-b)]. \tag{3.12}$$

This remark completes the proof of the theorem. We may choose the arbitrary function

ψ in (A10) so that the scalar b appearing in (3.5) vanishes. With this simplification we find that (see A15)

$$H = na \qquad n = 1, 2, 3. \tag{3.13}$$

The cosmological constants λ_f and λ_g depend on ϕ and u, hence the conformal factor ϕ is constant whenever one of the fields is the metric of an Einstein space. We shall now give some applications of the above theorem.

3.1. Coordinate transformations

General covariance of the f–g field theory requires simultaneous transformations of both fields under the general coordinate transformations (locking together property). Geometrically the two fields may be isometric to each other but the 'locking together property' prevents them from being identical. Utilising this fact and the above theorem, we propose a method to generate a solution of the empty f–g field equations with $\mathscr{L}^{\mathrm{I}}_{f\text{-}g}$, from an Einstein space. The first step is to choose an Einstein space $\tilde{g}_{\mu\nu}$ with cosmological constant $\tilde{\lambda}$. The second step is to perform a regular coordinate transformation. If the transformation matrix is $S^{\mu}{}_{\nu} = \partial x^{\mu}/\partial x'^{\nu}$, then the metric in the new coordinate system (x'^{μ}) is given by

$$g'^{\alpha\beta}(x') = S^{\alpha}{}_{\mu}S^{\beta}{}_{\nu}\tilde{g}^{\mu\nu}(x'). \tag{3.14}$$

The third step is to choose the new coordinates in such a way that (3.14) takes the form

$$\tilde{g}'_{\mu\nu}(x') = \phi^{-1}\bar{g}_{\mu\nu}(x') - \phi^{-1}(\phi + a)^{-1}H_{\mu\nu} \tag{3.15}$$

where ϕ, a and $H_{\mu\nu}$ obey the conditions (3.5), (3.6) and (3.13) and where $\bar{g}_{\mu\nu}(x')$ is also a metric of an Einstein space with cosmological constant $\tilde{\lambda}$. The final step is to identify $\tilde{g}'_{\mu\nu}(x')$ in (3.15) as the f field and $\bar{g}_{\mu\nu}(x')$ as the g field (see appendix 2, for application).

We divide the solutions obtainable by this method in two classes, according to whether the fields f and g are isometric or not. The solutions found by Salam and Strathdee (1977) and Isham and Storey (1978) fall into the first class. We shall show how to obtain this solution in appendix 2 by use of the method presented here. It is clear from (3.13) that there are at least three f fields corresponding to a g field. For all these solutions the cosmological constants are

$$\lambda_f = \tfrac{1}{4}M^2K^{u}[(\tfrac{1}{2} - u)I + J] \tag{3.16}$$

$$\lambda_g = \mu_0{}^2\tfrac{1}{4}M^2K^{u-1/2}(uI - J) \tag{3.17}$$

and

$$\begin{aligned} I &= -3(a+1)[1+(1-n)a] + n(1-n)a^2 \\ J &= \tfrac{1}{2}[3+(1-n)a][3+(3-n)a] \\ K &= -\tfrac{1}{8}J[(1-n)(n-3)a^2 - 9]. \end{aligned} \tag{3.18}$$

The Isham–Storey solution is obtained simply by taking the g field as the Schwarzschild–de Sitter metric and letting $n = 1$. As a further illustration of our method we present the following generalisation of the Isham–Storey solution

$$(\mathrm{d}S_g)^2 = q\,\mathrm{d}t^2 - q^{-1}\,\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2 \tag{3.19}$$

$$H_{\mu\nu}\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu} = \phi(3-\phi)(U_{\mu}\,\mathrm{d}x^{\mu})^2 - (2\phi - 3)(q_{\mu}\,\mathrm{d}x^{\mu})^2 \tag{3.20}$$

where

$$U_\mu \, dx^\mu = (q\phi^{-1} - \beta p)^{1/2} \, dt + q^{-1}\beta^{1/2}(q-p)^{1/2} \, dr \tag{3.21}$$

$$q_\mu \, dx^\mu = r \sin\theta \, d\phi \tag{3.22}$$

and the functions p and q are given in appendix 2, with

$$\beta = (3-\phi)^{-1} \qquad 0<\phi<3. \tag{3.23}$$

This solution corresponds to $n = 2$ and hence $a = 3 - 2\phi$, where ϕ remains an arbitrary constant. The other solutions, for example the solution with $n = 3$, may be found by using a similar procedure.

3.2. Conformal scaling

As a second application of the theorem we let $H_{\mu\nu} = 0$. The fields become conformally related and the cosmological constants are given simply by

$$\lambda_f = \tfrac{3}{2}M^2(\phi-1)\phi^{4u}(2u\phi+1-2u) \tag{3.24}$$

$$\lambda_g = -\tfrac{3}{2}M^2\mu_0^2(\phi-1)\phi^{4u-2}[(1+2u)\phi-2u] \tag{3.25}$$

since $f_{\mu\nu} = \phi^{-1}g_{\mu\nu}$ and ϕ is a constant, then

$$\lambda_f = \phi\lambda_g \tag{3.26}$$

hence ϕ is constrained to obey the following equation

$$2u\phi^2 + [1-2u+\mu_0^2(1+2u)]\phi - 2u\mu_0^2 = 0 \tag{3.27}$$

where we assumed that $\phi \neq 1$ (see Pirani (1971) for $\phi = 1$ case; this result corresponds to $u = 0$ in which case (3.27) has no solution for ϕ, hence $\phi = 1$). We note that ϕ in (3.27) has sometimes negative values, for example if

$$u = \frac{1}{2}\,\frac{1+\mu_0^2}{1-\mu_0^2}$$

then $\phi = \pm\mu_0$, hence there are some f–g fields having different signatures.

3.3. Generalised Kerr–Schild metrics

Let us assume that $H_{\mu\nu} = 2V\lambda_\mu\lambda_\nu$, where V is a scalar function and λ_μ is a null vector. In this case $H = a = 0$, $\phi = \frac{3}{2}$, $g/f = \phi^4$ and

$$\lambda_f = \tfrac{3}{4}M^2(\tfrac{3}{2})^{4u}(1+u) \tag{3.28}$$

$$\lambda_g = -\tfrac{3}{4}M^2\mu_0^2(\tfrac{3}{2})^{4u-2}(\tfrac{3}{2}+u) \tag{3.29}$$

and (3.8) reduces to

$$f_{\mu\nu} = \phi^{-1}(g_{\mu\nu} - 2V\lambda_\mu\lambda_\nu) \tag{3.30}$$

which is of the generalised Kerr–Schild type. If $g_{\mu\nu}$ is the metric of an Einstein space, we must find V and λ_μ so that $f_{\mu\nu}$ belongs to an Einstein space as well. The Carter (1972)–Plebanski–Demianski (1976) solution (see appendix 2) is of this form.

We present now one more new solution of this form. The solution is (Gürses and Güven 1980)

$$(\mathrm{d}S_g)^2 = \Omega^{-2}(2\,\mathrm{d}u\,\mathrm{d}v - 2\,\mathrm{d}z\,\mathrm{d}\bar{z}) \tag{3.31}$$

$$\lambda_\mu\,\mathrm{d}x^\mu = \mathrm{d}u \tag{3.32}$$

$$V = V(u, x, y) \qquad \Box_g V = 0 \tag{3.33}$$

where

$$\Omega = \Omega_0 + \alpha z + \bar{\alpha}\bar{z} \qquad z = \tfrac{1}{2}(x + \mathrm{i}y) \tag{3.34}$$

Ω_0 and α are arbitrary real and complex constants respectively, and $\lambda_g = 6\alpha\bar{\alpha}$, $\Box_g$ is the Beltrami operator of the second kind constructed by the field g. The special case $V = \Omega^3 h(u)$, where $h(u)$ is an arbitrary function of u, has recently been given by Barnes (1979). In this and Carter–Plebanski–Demianski solutions we have $\lambda_f = \phi\lambda_g$ as in the case of the conformally related f–g fields. Combination of this relation with (3.28) and (3.29) leads to

$$u = -\frac{1 + \mu_0^2}{1 + \frac{2}{3}\mu_0^2}.$$

We have given, so far, some solutions of the empty f–g field equations so that each of the fields is an Einstein space. Another solution, which is not constructed by means of our theorem, is obtained by letting $\phi = 1$ and using the generalised Kerr–Schild related form (3.30) in the field equations (3.1) and (3.2). Field equations simply reduce to

$$G_{\mu\nu}(f) = -M^2 V\lambda_\mu\lambda_\nu \tag{3.35}$$

$$G_{\mu\nu}(g) = \mu_0^2 M^2 V\lambda_\mu\lambda_\nu. \tag{3.36}$$

As far as the g field is concerned, which is a solution of (3.36), we may use a known solution of the Einstein field equations with a pure radiation source. Hence the problem is to find a function V and a null vector λ_μ which satisfy (3.35). A solution of this type has been given by Aichelburg *et al* (1971); it reads

$$\begin{aligned}&(\mathrm{d}S_g)^2 = 2\,\mathrm{d}u\,\mathrm{d}v - \mathrm{d}x^2 - \mathrm{d}y^2 - 2G(u, x, y)\,\mathrm{d}u^2\\ &\lambda_\mu\,\mathrm{d}x^\mu = \mathrm{d}u \qquad V = V(u, x, y)\end{aligned} \tag{3.37}$$

where G and V are subject to satisfying the following differential equations

$$\begin{aligned}&\nabla^2\Phi = 0 \qquad \Phi = V + 2G\\ &\nabla^2 V + m^2 V = 0 \qquad m^2 = \tfrac{1}{2}M^2(1 + \mu_0^2)\end{aligned} \tag{3.38}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The interesting feature of this type of empty solutions is that the total energy momentum tensor density given in (2.28) vanishes. If this is the case for a general mixing term, the solutions given above are also independent of the mixing terms. When the mass m in (3.38) vanishes the solution reduces to the well known plane-fronted waves in general relativity (Ehlers and Kundt 1962).

4. Non-vacuum *f–g* field equations

f–g field equations with source terms have been considered by several authors. A solution with a null leptonic energy momentum tensor has been given by Aichelburg

(1973). On the other hand particle-like solutions to gauge fields coupled to the f–g field equations have been studied by Sayed (1979). In connection with the quark confinement problem Salam and Strathdee (1978) have given an approximate solution to the massive scalar field coupled to the f–g field. We shall here assume that no leptonic matter exists and let the g field be the metric of an Einstein space. With this assumption and (2.28) the field equations are given by

$$G_{\mu\nu}(g) = \lambda_g g_{\mu\nu} \tag{4.1}$$

$$G_{\mu\nu}(f) = \lambda_f f_{\mu\nu} + K_f^2 T_{\mu\nu}(f, \text{hadrons}). \tag{4.2}$$

In spite of the source term in (4.2) we still have the relation (A5) and the constraints (3.5), (3.6) and (3.13).

We shall present here some solutions of the above field equations when the energy momentum tensor of hadrons is substituted by the energy momentum tensors of electromagnetic and of Yang–Mills fields.

4.1. *Electromagnetic fields*

The g field is the Schwarzschild–de Sitter metric and the f field is the charged-Schwarzschild–de Sitter (Reissner–Nordström) metric, so that

$$(\mathrm{d}S_g)^2 = \mathrm{d}t^2 - \mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2 - 2(m/r + \tfrac{1}{3}\lambda r^2)(\mathrm{d}t + \mathrm{d}r)^2 \tag{4.3}$$

$$H_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \frac{e^2}{4r^2}(\mathrm{d}t + \mathrm{d}r)^2 \qquad \phi = \tfrac{3}{2} \tag{4.4}$$

$$A_\mu\,\mathrm{d}x^\mu = (e/r)(\mathrm{d}t + \mathrm{d}r). \tag{4.5}$$

The cosmological constants λ_f and λ_g are the same as those given in (3.24) and (3.25), A_μ is the electromagnetic vector potential, and e is the electric charge.

4.2. *Yang–Mills field*

The simple generalisation of the previous solution is to multiply the electromagnetic potential A_μ by the constants β^a and admit this product as the potential of the Yang–Mills fields (Yasskin 1975). We note that Sayed's solution can be reduced to the abelian (Reissner–Nordström) solution given in § 4.1 by a proper gauge transformation. In addition to these trivial solutions we give a solution of the f–g field equations coupled to null Yang–Mills fields. The solution is given as

$$(\mathrm{d}S_g)^2 = \Omega^{-2}(2\,\mathrm{d}u\,\mathrm{d}v - \mathrm{d}x^2 - \mathrm{d}y^2) \tag{4.6}$$

$$H_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = 2V(u, x, y)\,\mathrm{d}u^2 \tag{4.7}$$

$$A^a_\mu\,\mathrm{d}x^\mu = A^a(u, x, y)\,\mathrm{d}u \tag{4.8}$$

where Ω is given in (3.34) and (4.9)

$$\Box_g V = (1/4\pi) K_f^2 \gamma_{ab} B^{a\mu} B^{b\nu} g_{\mu\nu} \tag{4.10}$$

with

$$B^a{}_\mu = A^a{}_{,\mu} - (3/\lambda_g)^{1/2} A^a \Omega^{-1} \Omega_{,\mu} \tag{4.11}$$

and

$$\Box_g A^a = \lambda_g A^a. \tag{4.12}$$

The cosmological constants are exactly the same as given in (3.28) and (3.29); γ_{ab} given in (4.10) is the metric of the N-dimensional Lie group. The group indices (Latin indices) run from one to N. This solution in flat space–time has been given by Coleman (1977) as an example to the non-abelian plane–Yang–Mills waves. The generalisation of this solution to general relativity has been given by Güven (1979) and Trautman (1980). Our solution reduces to their solution by letting $\lambda_g = 0$. We note that the gauge potential has a mass term which is proportional to the cosmological constant λ_g, i.e.

$$\Box_f A^{\mu}{}_a = m^2 A^{\mu}{}_a \tag{4.13}$$

where $m^2 = \frac{4}{3}\lambda_g$ and $\Box_f$ is the Beltrami operator of the second kind constructed from the f field.

5. Conclusion

We studied the f–g field equations and showed that the Einstein space solutions are independent of the mixing terms. This clarifies why the Schwarzschild–de Sitter metric emerges as the common solution of the f–g field equations with different mixing terms. We have introduced several techniques to solve the field equations. Utilising these techniques and the known solutions of the Einstein field equations with the cosmological constant we have found classes of solutions. Most of the known solutions of the f–g field equations are in these classes.

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Appendix 1

In this appendix we give some matrix identities which are used in the text. Any 4×4 matrix, say $\phi^{\mu}{}_{\nu}$, defined in (2.6), satisfies its secular equation

$$\begin{aligned}-6\phi^{\mu}{}_{\alpha}\phi^{\alpha}{}_{\beta}\phi^{\beta}{}_{\rho}\phi^{\rho}{}_{\nu} + 6\phi^{\alpha}{}_{\alpha}\phi^{\mu}{}_{\beta}\phi^{\beta}{}_{\rho}\phi^{\rho}{}_{\nu} + 3[\phi^{\alpha}{}_{\beta}\phi^{\beta}{}_{\alpha} - (\phi^{\alpha}{}_{\alpha})^2]\phi^{\mu}{}_{\gamma}\phi^{\gamma}{}_{\nu} \\ + [(\phi^{\alpha}{}_{\alpha})^3 - 3\phi^{\alpha}{}_{\alpha}\phi^{\beta}{}_{\sigma}\phi^{\sigma}{}_{\beta} + 2\phi^{\alpha}{}_{\beta}\phi^{\beta}{}_{\sigma}\phi^{\sigma}{}_{\alpha}]\phi^{\mu}{}_{\nu} = 6\delta^{\mu}{}_{\nu}\det\phi^{\alpha}{}_{\beta}\end{aligned} \tag{A1}$$

where

$$\begin{aligned}\det\phi^{\alpha}{}_{\beta} = g/f = \tfrac{1}{24}[(\phi^{\alpha}{}_{\alpha})^4 + 8\phi^{\alpha}{}_{\alpha}\phi^{\mu}{}_{\sigma}\phi^{\sigma}{}_{\beta}\phi^{\beta}{}_{\mu} - 6(\phi^{\alpha}{}_{\alpha})^2\phi^{\beta}{}_{\sigma}\phi^{\sigma}{}_{\beta} \\ + 3(\phi^{\beta}{}_{\sigma}\phi^{\sigma}{}_{\beta})^2 - 6\phi^{\alpha}{}_{\beta}\phi^{\beta}{}_{\rho}\phi^{\rho}{}_{\sigma}\phi^{\sigma}{}_{\alpha}].\end{aligned} \tag{A2}$$

The inverse of $\phi^{\mu}{}_{\nu}$ is found from (A1), provided $\det\phi^{\alpha}{}_{\beta}$ is different from zero, as

$$\begin{aligned}\rho^{\mu}{}_{\nu} = (\det\phi^{\rho}{}_{\gamma})^{-1}\{-\phi^{\mu}{}_{\alpha}\phi^{\alpha}{}_{\beta}\phi^{\beta}{}_{\nu} + \phi^{\alpha}{}_{\alpha}\phi^{\mu}{}_{\beta}\phi^{\beta}{}_{\nu} + \tfrac{1}{2}[\phi^{\alpha}{}_{\beta}\phi^{\beta}{}_{\alpha} - (\phi^{\alpha}{}_{\alpha})^2]\phi^{\mu}{}_{\nu} \\ + \tfrac{1}{6}[(\phi^{\alpha}{}_{\alpha})^2 - 3\phi^{\alpha}{}_{\alpha}\phi^{\beta}{}_{\sigma}\phi^{\sigma}{}_{\beta} + 2\phi^{\alpha}{}_{\beta}\phi^{\beta}{}_{\sigma}\phi^{\sigma}{}_{\alpha}]\delta^{\mu}{}_{\nu}\}.\end{aligned} \tag{A3}$$

This last equation enables us to write $f_{\mu\nu}$ in terms of $\phi^{\mu}{}_{\nu}$ and $g_{\mu\nu}$,

$$f_{\mu\nu}=\frac{f}{6g}[(\phi^{\alpha}{}_{\alpha})^3-3\phi^{\alpha}{}_{\alpha}\phi^{\beta}{}_{\sigma}\phi^{\sigma}{}_{\beta}+2\phi^{\alpha}{}_{\beta}\phi^{\beta}{}_{\sigma}\phi^{\sigma}{}_{\alpha}]g_{\mu\nu}+\frac{f}{2g}[\phi^{\alpha}{}_{\beta}\phi^{\beta}{}_{\alpha}-(\phi^{\alpha}{}_{\alpha})^2]\phi^{\sigma}{}_{\mu}g_{\sigma\nu}$$
$$+\frac{f}{g}\phi^{\alpha}{}_{\alpha}\phi^{\sigma}{}_{\beta}\phi^{\beta}{}_{\mu}g_{\sigma\nu}-\frac{f}{g}\phi^{\beta}{}_{\alpha}\phi^{\alpha}{}_{\sigma}\phi^{\sigma}{}_{\mu}g_{\beta\nu}. \tag{A4}$$

We now introduce a scalar function ϕ and a symmetric tensor $H^{\mu\nu}$ as (Gürses 1979, 1980)

$$f^{\mu\nu}=\phi g^{\mu\nu}+H^{\mu\nu}; \tag{A5}$$

hence, we obtain

$$\frac{g}{f}=\phi^4+\phi^3H+\tfrac{1}{2}\phi^2H^2+\tfrac{1}{6}\phi H^3+\tfrac{1}{24}H^4-\tfrac{1}{4}(2\phi^2+2\phi H+H^2)h_2$$
$$+\tfrac{1}{8}h_2{}^2+\tfrac{1}{3}(\phi+H)h_3-\tfrac{1}{4}h_4 \tag{A6}$$

and

$$f_{\mu\nu}=f_0g_{\mu\nu}+f_1H_{\mu\nu}+f_2H_{\mu\alpha}H^{\alpha}{}_{\nu}+f_3H_{\mu\alpha}H^{\alpha\beta}H_{\beta\nu} \tag{A7}$$

where the indices of $H^{\mu\nu}$ are raised and lowered by $g^{\mu\nu}$ and $g_{\mu\nu}$ and

$$\begin{aligned} f_0&=(f/g)[\phi^3+\phi^2H+\tfrac{1}{2}\phi H^2-\tfrac{1}{2}(\phi+H)h_2+\tfrac{1}{6}H^3+\tfrac{1}{3}h_3]\\ f_1&=(f/g)(\tfrac{1}{2}h_2-\phi^2-\phi H-\tfrac{1}{2}H^2)\\ f_2&=(f/g)(\phi+H)\qquad f_3=-f/g \end{aligned} \tag{A8}$$

and

$$\begin{aligned} h_1&=H=H^{\mu}{}_{\mu}\qquad h_2=H^{\alpha}{}_{\beta}H^{\beta}{}_{\alpha}\\ h_3&=H^{\alpha}{}_{\beta}H^{\beta}{}_{\rho}H^{\rho}{}_{\alpha}\qquad h_4=H^{\alpha}{}_{\beta}H^{\beta}{}_{\sigma}H^{\sigma}{}_{\rho}H^{\rho}{}_{\alpha}. \end{aligned} \tag{A9}$$

We note that the form $f_{\mu\nu}$ given in (A5) is invariant under the following transformations

$$\begin{aligned} \phi&\to\tilde{\phi}=\phi+\psi\\ H^{\mu\nu}&\to\tilde{H}^{\mu\nu}=H^{\mu\nu}-\psi g^{\mu\nu} \end{aligned} \tag{A10}$$

where ψ is an arbitrary scalar function. It is clear that (A1) is also valid for $H^{\mu}{}_{\nu}$ when $\phi^{\mu}{}_{\nu}$ is replaced by $H^{\mu}{}_{\nu}$. If it satisfies an additional equation, such as

$$H^{\mu}{}_{\alpha}H^{\alpha}{}_{\nu}=aH^{\mu}{}_{\nu}+b\delta^{\mu}{}_{\nu} \tag{A11}$$

then the scalars a and b are found as

$$\begin{aligned} h_2&=aH+4b\\ h_3&=a^2H+4ab+bH\\ h_4&=a^3H+4a^2b+2abH+4b^2 \end{aligned} \tag{A12}$$

and hence, using (A11) in the secular equation of $H^{\mu}{}_{\nu}$, we obtain (Gürses 1979)

$$(H^2-3aH+3a^2-h_2)(H-2a)=0. \tag{A13}$$

Let us choose the function ψ in (A10) so that b in (A11) vanishes; then (A13) becomes

$$(H-a)(H-2a)(H-3a)=0 \tag{A14}$$

or

$$H=na \qquad n=1,2,3. \tag{A15}$$

Appendix 2

As an application of the method given in § 3, we give in this appendix two examples.

(i) *Isometric f–g fields.* First we choose the de Sitter–Schwarzschild metric

$$\begin{aligned}(\mathrm{d}S_{\tilde{g}})^2 &= \tilde{g}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \\ &= p\,\mathrm{d}t^2 - p^{-1}\,\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2\end{aligned} \tag{A16}$$

where

$$p = 1 - \frac{2m}{r} - \frac{\tilde{\lambda}}{3}r^2 \tag{A17}$$

and $\tilde{\lambda}$ is the cosmological constant, then we perform the following coordinate transformations

$$t = t' + G(r', t') \qquad r = \phi^{-1/2}r' \tag{A18}$$

where $G(r', t')$ is an arbitrary function of r' and t'. The transformed metric in (A16) becomes

$$(\mathrm{d}S_{\tilde{g}'})^2 = \Omega\bar{g}_{\mu\nu}\,\mathrm{d}x'^\mu\,\mathrm{d}x'^\nu - \Phi H_{\mu\nu}\,\mathrm{d}x'^\mu\,\mathrm{d}x'^\nu \tag{A19}$$

where

$$x'^\mu = (t', r', \theta, \phi) \qquad \Omega = \phi^{-1}$$

and

$$\begin{aligned}\Phi H_{t't'} &= -p'(1+G_{t'})^2 + q\phi^{-1} \\ \Phi H_{r'r'} &= -p'G_{r'}{}^2 + \phi^{-1}(q^{-1}+p'-1) \\ \Phi H_{r't'} &= -p'G_{r'}(1+G_{t'})\end{aligned} \tag{A20}$$

with

$$\Phi = \frac{1}{\phi(\phi+a)} \qquad G_{r'} = \frac{\partial G}{\partial r'} \qquad G_{t'} = \frac{\partial G}{\partial t'}$$

$$p' = 1 - \frac{2m\phi^{1/2}}{r'} - \frac{\tilde{\lambda}}{3}\phi^{-1}r'^2$$

$$q = 1 - \frac{2\mu}{r'} - \frac{\lambda}{3}r'^2$$

and the metric $\bar{g}_{\mu\nu}$ is given by

$$\bar{g}_{\mu\nu}\,\mathrm{d}x'^\mu\,\mathrm{d}x'^\nu = q\,\mathrm{d}t'^2 - q^{-1}\,\mathrm{d}r'^2 - r'^2\,\mathrm{d}\Omega^2. \tag{A21}$$

We notice that $\bar{g}_{\mu\nu}$ is also a de Sitter–Schwarzschild metric with mass μ and cosmological constant λ. Now the next step is to choose the arbitrary function $G(r', t')$ in (A18) so that $H_{\mu\nu}$ in (A20) satisfies the constraint (3.5c) and (3.13). We obtain

$$G_{r'}^2 = \frac{p'-q}{p'^2 q^2}(\beta p' - \tfrac{2}{3}q) \qquad G_{t'} = \beta$$

with $\phi = \frac{3}{2}$, $a = \beta^{-1} - \frac{3}{2}$, and β an integration constant, hence (A19) becomes

$$\begin{aligned}(\mathrm{d}S_{\bar{g}'})^2 &= \bar{g}'_{\mu\nu}\,\mathrm{d}x'^{\mu}\,\mathrm{d}x'^{\nu} \\ &= \tfrac{2}{3}\bar{g}_{\mu\nu}\,\mathrm{d}x'^{\mu}\,\mathrm{d}x'^{\nu} - \tfrac{2}{3}\beta(u_\mu\,\mathrm{d}x'^{\mu})^2\end{aligned} \tag{A22}$$

where

$$u_\mu\,\mathrm{d}x'^{\mu} = \frac{1}{q}[\tfrac{3}{2}(q-p')]^{1/2}\,\mathrm{d}r' + \left[\frac{3}{2\beta}(-p'\beta + \tfrac{2}{3}q)\right]^{1/2}\mathrm{d}t'. \tag{A23}$$

As a final step we identify $\bar{g}'_{\mu\nu}$ as the f field and $\bar{g}_{\mu\nu}$ as the g field. The cosmological constants are given by

$$\begin{aligned}\lambda_f &= \tilde{\lambda} = \tfrac{3}{4}M^2(\tfrac{3}{2})^{3u}\beta^{-u-1}[1+(1-\tfrac{1}{2}\beta)(u-\tfrac{1}{2})] \\ \lambda_g &= \lambda = -\tfrac{3}{4}\mu_0{}^2M^2(\tfrac{3}{2})^{3u-3/2}\beta^{-u-1/2}[1+u(1-\tfrac{1}{2}\beta)].\end{aligned} \tag{A24}$$

If $\beta = \frac{2}{3}$ then $a = 0$ and u_μ becomes a null vector.

(ii) *Anisometric f–g fields.* We start with the metric given by Carter (1972) and Plebanski and Demianski (1976) (Kerr–de Sitter solution)

$$\begin{aligned}(\mathrm{d}S_{\bar{g}}) = {}&\frac{\Delta}{\Sigma\varepsilon_0{}^2}(\mathrm{d}t - \rho_0 a\sin^2\theta\,\mathrm{d}\theta)^2 - \sin^2\theta\frac{\Lambda}{\Sigma\varepsilon_0{}^2}[a\,\mathrm{d}t - \rho_0(r^2+a^2)\,\mathrm{d}\phi]^2 \\ &- \rho_0{}^2\left(\frac{\Sigma}{\Delta}\mathrm{d}r^2 + \frac{\Sigma}{\Lambda}\mathrm{d}\theta^2\right)\end{aligned} \tag{A25}$$

where

$$\begin{aligned}\Sigma &= r^2 + a^2\cos^2\theta \\ \Delta &= (r^2+a^2)(1-\tfrac{1}{3}\tilde{\lambda}r^2) - 2mr \\ \Lambda &= 1 + \tfrac{1}{3}\tilde{\lambda}a^2\cos^2\theta \\ \varepsilon_0 &= 1 - \tfrac{1}{3}\tilde{\lambda}a^2\end{aligned} \tag{A26}$$

and ρ_0 is an arbitrary constant which may be eliminated by a coordinate transformation. Now we perform the following coordinate transformations

$$\mathrm{d}t = \varepsilon_0\rho_0\,\mathrm{d}t' + \left(\frac{r^2+a^2}{\Delta} - \frac{1}{1-\frac{1}{3}\tilde{\lambda}r^2}\right)\mathrm{d}r$$

$$\mathrm{d}\phi = \varepsilon_0\left[\mathrm{d}\phi' + \tfrac{1}{3}\tilde{\lambda}a\frac{1}{1+\frac{1}{3}\tilde{\lambda}a^2}\mathrm{d}t' + a\left(\frac{1}{\Delta} - \frac{1}{(r^2+a^2)(1-\frac{1}{3}\tilde{\lambda}r^2)}\right)\mathrm{d}r\right].$$

Then we obtain

$$(\mathrm{d}S_{\bar{g}'})^2 = \rho_0{}^2\left((\mathrm{d}S_{\bar{g}})^2 - \frac{2mr}{\Sigma}(\lambda_\mu\,\mathrm{d}x'^{\mu})^2\right) \tag{A27}$$

where

$$(\mathrm{d}S_{\bar{g}})^2 = (1-\tfrac{1}{3}\tilde{\lambda}r^2)\frac{\Lambda}{1+\frac{1}{3}\tilde{\lambda}a^2}\,\mathrm{d}t'^2 - \frac{\Sigma}{(r^2+a^2)(1-\frac{1}{3}\tilde{\lambda}r^2)}\,\mathrm{d}r^2$$

$$-\frac{\Sigma}{\Lambda}\,\mathrm{d}\theta^2 - (r^2+a^2)(1+\tfrac{1}{3}\tilde{\lambda}a^2)\sin^2\theta\,\mathrm{d}\phi^2 \qquad \text{(A28)}$$

$$\lambda_\mu\,\mathrm{d}x'^\mu = \frac{\Lambda}{1+\frac{1}{3}\tilde{\lambda}a^2}\,\mathrm{d}t' + \frac{\Sigma}{(r^2+a^2)(1-\frac{1}{3}\tilde{\lambda}r^2)}\,\mathrm{d}r + a\sin^2\theta\,\mathrm{d}\phi' \qquad \text{(A29)}$$

and we observe that the trace of $H_{\mu\nu}$ vanishes. Hence

$$H_{\mu\nu} = \frac{2mr}{\Sigma}\lambda_\mu\lambda_\nu \qquad \lambda^\mu\lambda_\mu = 0; \qquad \text{(A30)}$$

then the condition (3.5*c*) is automatically satisfied by letting $\rho_0{}^2 = \phi^{-1} = \frac{2}{3}$. We can now identify the metric $\tilde{g}_{\mu\nu}$ as the f field and the metric $\bar{g}_{\mu\nu}$ as the g field. The metric $\bar{g}_{\mu\nu}$ is the de Sitter metric written in a different coordinate system (see Carter 1972). The cosmological constants are

$$\lambda_f = \tilde{\lambda} \qquad \lambda_g = \tfrac{2}{3}\tilde{\lambda} \qquad \text{(A31)}$$

which leads to

$$u = -\frac{1+\mu_0{}^2}{1+\frac{2}{3}\mu_0{}^2} \qquad \text{(A32)}$$

where μ_0 is defined in (3.4).

References

Aichelburg P C 1973 *Phys. Lett.* **44B** 370
Aichelburg P C, Mansouri R and Urbantke H K 1971 *Phys. Rev. Lett.* **27** 1533
Barnes A 1979 *J. Phys. A: Math. Gen.* **12** 1493
Carter B 1973 in *Black Holes, 1972 Les Houches Lectures* ed. C M DeWitt and B S DeWitt (New York: Gordon and Breach) pp 57–125
Coleman S 1977 *Phys. Lett.* **70B** 59
Ehlers J and Kundt W 1962 in *Gravitation: An Introduction to Current Research* ed. L Witten (New York: Wiley) pp 49–101
Gürses M 1979 *Phys. Rev.* D **20** 1019
—— 1980 *J. Phys. A: Math. Gen.* **13** L223
Gürses M and Güven R 1980 *Generalized Kerr–Schild Metrics* unpublished
Güven R 1979 *Phys. Rev.* D **19** 471
Isham C J, Salam A and Strathdee J 1971 *Phys. Rev.* D **3** 867
Isham C J and Storey D 1978 *Phys. Rev.* D **18** 1047
Lee T D, Kroll N M and Zumino B 1967 *Phys. Rev.* **157** 212
Pirani F A E *ICTP Report* No IC(71/144) unpublished
Plebanski J and Demianski M 1976 *Ann. Phys., NY* **98** 98
Salam A and Strathdee J 1977a *Phys. Lett.* **67B** 429
—— 1977b *Phys. Rev.* D **16** 2268
—— 1978 *Phys. Rev.* D **18** 4596
Sayed W A 1979 *Phys. Rev.* D **20** 1465
Trautman A 1980 *J. Phys. A: Math. Gen.* **13** L1
Yasskin P B 1975 *Phys. Rev.* D **12** 2212